THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MMAT5540 Advanced Geometry 2016-2017
Supplementary Exercise 3

In the following exercises, we assume the axioms of incidence, axioms of betweenness, axioms of congruence for line segments and angles.

1. Let $\tilde{\mathcal{P}}$ be the set of all oriented line segments and let $\overrightarrow{AB}, \overrightarrow{CD}, \overrightarrow{EF} \in \tilde{\mathcal{P}}$. Prove that
$$(\overrightarrow{AB} + \overrightarrow{CD}) + \overrightarrow{EF} = \overrightarrow{AB} + (\overrightarrow{CD} + \overrightarrow{EF}).$$

2. Recall that if AB, CD are line segments, AB is less than CD if there exists E such that $C * E * D$ and $AB \cong CE$. We denote it by $AB < CD$.

   However, this definition depends on the choice of orientation of CD (but not AB, since $AB \cong BA$).

   Show that the above definition is well established by proving that $AB < CD$ if and only if $AB < DC$.

3. Let AB be a line segment and let l be a line. Prove that there exists a sequence of points C_n such that $C_n * C_{n+1} * C_{n+2}$ and $C_nC_{n+1} \cong AB$ for all $n = 1, 2, 3, \ldots$.

4. Let $\angle BAC$ and $\angle BAD$ be supplementary angles and $\angle BAC \cong \angle B'A'C'$.

   Prove that $\angle B'A'C'$ and $\angle B'A'D'$ are supplementary angles if and only if $\angle BAD \cong \angle B'A'D'$.

5. Prove that any two right angles are congruent to each other.

6. Rephase proposition 5 and 6 in Book I of Eucild's Elements and rewrite the proofs of them.

Lecturer's comment:

1. There exists a unique G such that $A * B * G$ and $BG \cong CD$ and $\overrightarrow{AB} + \overrightarrow{CD} = \overrightarrow{AG}$.

   Also, there exists a unique H such that $A * G * H$ and $GH \cong EF$ and $(\overrightarrow{AB} + \overrightarrow{CD}) + \overrightarrow{EF} = \overrightarrow{AG} + \overrightarrow{EF} = \overrightarrow{AH}$.

   On the other hand, there exists a unique I such that $C * D * I$ and $DI \cong EF$ and $\overrightarrow{CD} + \overrightarrow{EF} = \overrightarrow{CI}$.

   Also, there exists a unique J such that $A * B * J$ and $BJ \cong CI$ and $\overrightarrow{AB} + (\overrightarrow{CD} + \overrightarrow{EF}) = \overrightarrow{AB} + \overrightarrow{BJ} = \overrightarrow{AJ}$.

   Then, $BG \cong CD$ and $GH \cong EF \cong DI$, by axiom **C3**, $BH \cong CI$. Also $CI \cong BJ$, so $BH \cong BJ$.

   Since J, H and A are on the opposite side of B, J and H are on the same side of B. By axiom **C1**, $H = J$. As a result,
$$(\overrightarrow{AB} + \overrightarrow{CD}) + \overrightarrow{EF} = \overrightarrow{AB} + (\overrightarrow{CD} + \overrightarrow{EF}).$$

2. Suppose that $AB < CD$, there exists E such that $C * E * D$ and $AB \cong CE$.

   By axiom **C1**, there exists unique E' on the ray r_{DC} such that $AB \cong DE'$. We claim that $D*E'*C$, and so $AB < DC$.

   Suppose not, then we have $C = E'$ or $E' * C * D$.

   (i) If $C = E'$, let F be a point such that F and E are on the opposite side of D, and $DF \cong DE$ (axiom **C1**).

   Since $CE \cong AB \cong CD$ and $ED \cong DF$, by axiom **C3** $CD \cong CF$. Note that D and F are on the same side of C, it forces that $C = F$ by axiom **C1**, which is a contradiction.

   (ii) If $E'*C*D$, let F be a point such that F and E are on the opposite side of D , and $DF \cong DE$ (axiom **C1**).

   Furthermore, let G be a point such that G and D are on the opposite side of F, and $FG \cong E'C$ (axiom **C1**).

   Since $CE \cong AB \cong E'D$ and $ED \cong DF$, by axiom **C3** $CD \cong E'F$. Again, $CD \cong E'F$ and $E'C \cong FG$, by axiom **C3** $E'D \cong E'G$. Note that D and F are on the same side of E', it forces that $C = F$ by axiom **C1**, which is a contradiction.

3. Let AB be a line segment and let l be a line.

   By axiom **I2**, there exists 2 points lying on l. We choose any one of them and call it C_1

   Take one ray $r \subset l$ originated from C_1, by axiom **C1**, there exists C_2 on r such that $C_1C_2 \cong AB$.

   By axiom **C1**, there exists C_3 such that C_3 and C_1 are on the opposite side of C_2 and $C_2C_3 \cong AB$.

   Repeating this process, we can obtain a sequence of points C_n as required.

4. (i) "$\Rightarrow$": By choosing another B', C' and D' if necessary, we may assume $AB \cong A'B'$, $AC \cong A'C'$ and $AD \cong A'D'$.

   $AB \cong A'B'$, $AC \cong A'C'$ and $\angle BAC \cong \angle B'A'C'$ implies that $BC \cong B'C'$ and $\angle BCA \cong \angle B'C'A'$ (axiom **C6**, SAS).

   Then, we have $CA \cong C'A'$ and $AD \cong A'D'$, so $CD \cong C'D'$ (axiom **C3**.

   $CD \cong C'D'$, $BC \cong B'C'$ and $\angle BCA \cong \angle B'C'A'$ implies that $BD \cong B'D'$ and $\angle BDA \cong \angle B'D'A'$ (axiom **C6**, SAS).

   $BD \cong B'D'$, $AD \cong A'D'$ and $\angle BDA \cong \angle B'D'A'$ implies that $\angle BAD \cong \angle B'A'D'$ (axiom **C6**, SAS).

   (ii) "$\Leftarrow$": Let E' be a point such that E' lies on the line $l_{A'C'}$, and E', C' are on the opposite side of A'. Then $\angle B'A'C'$ and $\angle B'A'E'$ are supplementary.

   By the previous part, we have $\angle BAD \cong \angle B'A'E'$. By assumption, we have $\angle BAD \cong \angle B'A'D'$, so $\angle B'A'D' \cong \angle B'A'E$.

   By axiom **C4**, D' lies on the ray $r_{A'E'}$ and so $\angle B'A'C'$ and $\angle B'A'D'$ are supplementary angles.

5. Suppose that α and α' are right angles. By assumption, $\alpha \cong \beta$ and $\alpha' \cong \beta'$. We claim that $\alpha \cong \alpha'$.

   Suppose the contrary, without loss of generality, let $\alpha < \alpha'$.

Then there exists a ray $r_{A'E'}$ in the interior of α' such that $\angle E'A'B' \cong \alpha$.

Claim: The ray $r_{A'C'}$ is in the interior of $\angle E'A'D'$, and so $\beta' < \angle E'A'D'$.

By crossbar theorem, the ray $r_{A'E'}$ intersect the line segment at a point F'. Therefore, C' and B' are on the opposite side of the line $l_{A'E'}$. Also, D' and B' are on the opposite side of the line $l_{A'E'}$. Therefore, C' and D' are on the same side of the line $l_{A'E'}$.

On the other hand, we have $C' * F' * B'$ and so C' and F' are on the same side of $l_{A'D'}$. Also, E' and F' are on the same side of $l_{A'D'}$. Therefore, C' and E' are on the same side of the line $l_{A'D'}$.

Therefore, the ray $r_{A'C'}$ is in the interior of $\angle E'A'D'$.

Note that $\angle E'A'B' \cong \alpha$, α, β and $\angle E'A'B'$, $\angle E'A'D'$ are supplementary, so $\angle E'A'D' \cong \beta$.

As a result, $\alpha' \cong \beta' < \angle E'A'D'' \cong \beta \cong \alpha$ which contradicts to that $\alpha < \alpha'$.

6. (a) (Proposition 5) Given an isosceles triangle ABC with $AB \cong AC$, then $\angle ABC \cong \angle ACB$ (known as "base $\angle$s, isos. $\triangle$). Furthermore, if D and E are points such that $A * B * D$ and $A * C * E$, then $\angle CBD \cong \angle BCE$.

proof: Take a point F on the ray r_{BD}. By axiom (**C.1**), there exists a unique point G on the ray r_{AE} such that $AF \cong AG$. Since $AB \cong AC$, $AG \cong AF$ and $\angle BAG \cong \angle CAF$, by axiom (**C.6**), we have $\angle ABG \cong \angle ACF$, $\angle AGB \cong \angle AFC$ and $BG \cong CF$. By construction, we have $A * B * F$ and $A * C * G$, also $AF \cong AG$ and $AB \cong AC$, so $BF \cong CG$. Then, since $BF \cong CG$, $\angle BFC \cong \angle CGB$ and $CF \cong BG$, by axiom (**C.6**), we have $\angle FBC \cong \angle GCB$, $\angle BCF \cong \angle CBG$. Again $A * B * F$ and $A * C * G$ implies r_{CB} and r_{BC} are in the interior of $\angle ACF$ and $\angle ABG$ respectively, also $\angle ABG \cong \angle ACF$ and $\angle CBG \cong \angle BCF$, so $\angle ABC \cong \angle ACB$.

(b) (Proposition 6) Given a triangle ABC. If $\angle ABC \cong \angle ACB$, then $AB \cong AC$ (known as "base $\angle$s equal").

proof: Assume that AB is not congruent to AC, then either $AB > AC$ or $AB < AC$. Without loss of generality, we assume $AB > AC$. By definition, there exists a point D such that $A * D * B$ and $BD \cong AC$. $DB \cong AC$, $BC \cong CB$ and $\angle DBC \cong \angle ACB$ (assumption), by axiom (**C.6**), we have $\angle DBC \cong \angle ACB$ and $\angle DCB \cong \angle ABC = \angle DBC$. Therefore, $\angle ACB \cong \angle BCD$. By axiom (**C.6**), there is only one angle on the same side of the ray r_{CD} congruent to $\angle ACB$, so $r_{CA} = r_{CD}$ and $A = D$, which contradicts to that $A * D * B$.